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STABILITY OF STATIONARY SOLUTIONS OF THE SYSTEM OF EQUATIONS OF THE COMBUSTION THEORY *

V. S. BAUSHEV, V. N. VILIUNOV, and A. M. TIMOKHIN

Local stability of stationary solutions of the system of equations of the combustion theory is investigated using a thermal diffusion model. The problem is reduced to the analysis of the point spectrum of a differential operator determined on vector functions in an unbounded region. A method is proposed for determining the point spectrum. The spectral region for any arbitrary Lewis number is determined in the stability region. It is shown that in the case of the Lewis number equal unity there are no spectral points outside that region.

1. Statement of the problem. The heat diffusion mechanism of flame propagation over a homogeneous combustible mixture is defined by the system of equations

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - \Phi(U, V), \quad \frac{\partial V}{\partial t} = L\frac{\partial^2 V}{\partial x^2} - \Phi(U, V) \tag{1.1}$$
$$(-\infty < x < \infty, \ t > 0)$$

where U and V are the dimensionless temperature and concentration, and L is the Lewis number. The relation between dimensionless and dimensional quantities was defined in /1/. In the case of first order reactions the heat release function is of the form

$$\Phi(U, V) = Vf(U)$$

where $f(U)$ is assumed to be a fairly smooth function that satisfies the conditions

$$f(U) > 0, \quad U < \varepsilon; \quad f(U) \equiv 0, \quad U \geqslant \varepsilon; \quad 0 < \varepsilon < 1 \tag{1.2}$$

For definiteness we set $f(0) = 1$. The meaning of the cut-off parameter is the same as in /2/.

It was shown in /3/ that system (1.1) has the stationary solution $u(\xi), v(\xi)$ of the form of a running wave that satisfies the system of equations

$$\frac{d^2u}{d\xi^2} + \omega\frac{du}{d\xi} - \Phi(u, v) = 0, \quad u(-\infty) = v(-\infty) = 0 \tag{1.3}$$
$$L\frac{d^2v}{d\xi^2} + \omega\frac{dv}{d\xi} - \Phi(u, v) = 0, \quad u(\infty) = v(\infty) = 0$$

where $\xi = x - \omega t$, and ω is the wave propagation velocity.

The problem of investigating the stability of stationary solutions of system (1.1) under small perturbations is formulated as follows. Let the perturbed solutions which, as $|x| \to \infty$, satisfy the same conditions as the stationary ones, be defined by

$$U = u(\xi) + \delta\varphi + \delta^2\varphi_1 + \ldots, \quad V = v(\xi) + \delta\psi + \delta^2\psi_1 + \ldots \tag{1.4}$$

where δ is the small parameter.

Substituting (1.4) into (1.1) and using the expansion of $f(U)$ in a series in powers of δ, we obtain linearized equations in φ and ψ

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial\xi^2} + \omega\frac{\partial\varphi}{\partial\xi} + a\varphi - b\psi \tag{1.5}$$
$$\frac{\partial\psi}{\partial t} = L\frac{\partial^2\psi}{\partial\xi^2} + \omega\frac{\partial\psi}{\partial\xi} + a\varphi - b\psi, \quad \varphi, \psi|_{|\xi|\to\infty} = 0$$
$$a = -\frac{\partial\Phi}{\partial u} = -vf'(u), \quad b = \frac{\partial\Phi}{\partial v} = f(u)$$

Here and subsequently a prime denotes differentiation with respect to the variable u. Passing from the stationary system of coordinates (x, t) to coordinates (ξ, t) attached to the flame results in the coefficients in Eqs. (1.5) becoming independent of t.

We seek a solution of system (1.5) of the form

$$\varphi = z_1(\xi, \lambda)e^{-\lambda t}, \quad \psi = z_2(\xi, \lambda)e^{-\lambda t} \ (\lambda = \lambda_r + i\lambda_i) \tag{1.6}$$

Substituting (1.6) into (1.5) we obtain

$$\frac{d^2z_1}{d\xi^2} + \omega\frac{dz_1}{d\xi} + az_1 - bz_2 = -\lambda z_1 \tag{1.7}$$
$$L\frac{d^2z_2}{d\xi^2} + \omega\frac{dz_2}{d\xi} + az_1 - bz_2 = -\lambda z_2, \quad z_1, z_2|_{|\xi|\to\infty} = 0$$

*Prikl.Matem.Mekhan.,44,No.1,96-103,1980 or briefly $\mathbf{A}z = -\lambda z$, where z is a vector with components z_1 and z_2.

We have stability when $\lambda_r > 0$. The investigation of stability is, thus reduced to the analysis of the point spectrum (eigenvalues) of operator $\mathbf{A}$.

Such method of stability investigation differs from the method which has been used for a considerable time in hydrodynamics /4/, and which was recently proved by that the operator $\mathbf{A}$ is determined on functions in an unbounded region. A similar approach to stability investigation was, apparently, first used in /6/.

Note that $\lambda = 0$ is an eigenvalue of operator $\mathbf{A}$. The eigen functions related to this value of λ

$$z_1(\xi, 0) = \frac{du}{d\xi}, \quad z_2(\xi, 0) = \frac{dv}{d\xi} \tag{1.8}$$

corresponds to solutions of the shift type (see expansions (1.4))

$$\lim U = u(\xi + \delta), \quad \lim V = v(\xi + \delta), \quad t \to \infty$$

2. Subsidiary expressions.

The introduction of a new independent variable u and of new function $p(u) = du/d\xi$ transforms system (1.3) to the form /1/

$$pp' + \omega p = \Phi(u, v) \tag{2.1}$$
$$Lpv' + \omega(v - u) = p, \quad 0 < u < 1$$
$$p(0) = v(0) = p(1) = 0$$

In the neighborhood of $u = 0$ we have the following expansions in series in powers of u

$$p = p_1 u + p_2 u^2 + \ldots, \quad \Phi = \Phi_1 u + \Phi_2 u^2 + \ldots, \tag{2.2}$$
$$v = v_1 u + v_2 u^2 + \ldots$$

where the slope coefficients p_1, Φ_1, and v_1 (assuming that $f(0) = 1$) are of the form

$$p_1 = \frac{2}{\omega + \sqrt{\omega^2 + 4L}}, \quad v_1 = \Phi_1 = p_1^2 + \omega p_1 \tag{2.3}$$

In the region of $0 < u < \varepsilon$ we have in conformity with (1.2)

$$\Phi(u, v) \equiv 0, \quad p = \omega(1 - u) \tag{2.4}$$

In the subsequent analysis of solutions of system (1.7) we shall need the following results. Let us assume the existence of the quadratic equation in γ (with ν a real number)

$$\gamma^2 + \nu\gamma + \beta = 0 \quad (\beta = \beta_r + i\beta_i)$$

When $\nu < 0$ γ_r has always on positive value. The condition of positiveness of the other is

$$\beta_r > \beta_i^2/\nu^2 \tag{2.5}$$

and when $\nu > 0$ γ_r has always one negative value. The condition of positiveness of the other is

$$\beta_r < \beta_i^2/\nu^2 \tag{2.6}$$

3. Analysis of integral curves of system (1.7).

We take function $u(\xi)$ which is the solution of the stationary problem, as the new independent variable. System (1.7) then assumes the form

$$p^2 z_1'' + p(p' + \omega) z_1' + (a + \lambda) z_1 - b z_2 = 0 \tag{3.1}$$
$$Lp^2 z_2'' + p(Lp' + \omega) z_2' + a z_1 + (\lambda - b) z_2 = 0$$

The idea of this investigation is to analyze the solution of system (3.1) in the neighborhood of $u = 0$ and in region $\varepsilon < u < 1$ in which, taking into account (2.4), we present system (3.1) in the form

$$\omega^2 (1 - u)^2 z_1'' + \lambda z_1 = 0$$
$$L\omega^2 (1 - u)^2 z_2'' + \omega^2 (1 - L)(1 - u)^2 z_2' + \lambda z_2 = 0$$

whose general solution is

$$z_1 = c_1 (1 - u)^{D_1} + C_2 (1 - u)^{D_2}, \quad D_{1r} > D_{2r} \tag{3.2}$$
$$z_2 = C_3 (1 - u)^{B_1} + C_4 (1 - u)^{B_2}, \quad B_{1r} > B_{2r}$$

where D_k and B_k $(k = 1, 2)$ are the roots of the characteristic equations

$$D^2 - D + \lambda/\omega^2 = 0, \quad LB^2 - B + \lambda/\omega^2 = 0 \tag{3.3}$$

where C_k and C_{k+2} are constants of integration. Investigation of the case of multiple roots of Eqs. (3.3) does not present difficulties, since

$$z_1 = C_1 (1 - u)^{\frac{1}{2}} + C_2 (1 - u)^{\frac{1}{2}} \ln(1 - u), \quad \lambda = \frac{\omega^2}{4}, \qquad z_2 = C_3 (1 - u)^{\frac{1}{2L}} + C_4 (1 - u)^{\frac{1}{2L}} \ln(1 - u), \quad \lambda = \frac{\omega^2}{4L}$$

If the four-parameter set of curves (3.2) is to emerge form $u = 1$, it is necessary, in conformity with (2.5), to stipulate

$$\lambda_r > \lambda_i^2/\omega^2, \quad \lambda_r > L\lambda_i^2/\omega^2 \tag{3.4}$$

Let us now examine the behavior of solutions of system (3.1) in the neighborhood of point $u = 0$. Taking into account equalities (2.2) and (2.3) we find that the point $u = 0$ is for (3.1) a regular singular point /7/. We seek the solution of system (3.1) in the neighborhood of $u = 0$ of the form

$$z_1 = \sum_{k=0}^{\infty} z_{1k} u^{k+\rho}, \quad z_2 = \sum_{k=0}^{\infty} z_{2k} u^{k+\rho} \tag{3.5}$$

Substituting (3.5) into (3.1) and equating to zero the coefficients at u^ρ, we obtain for the coefficients z_{10} and z_{20} the system

$$\begin{aligned}(p_1^2\rho^2 + \omega p_1\rho + \lambda) z_{10} - z_{20} &= 0 \\ (Lp_1^2\rho^2 + \omega p_1\rho + \lambda - 1) z_{20} &= 0\end{aligned} \tag{3.6}$$

To obtain a nontrivial solution of system (3.6) it is necessary to equate to zero the expressions in parentheses in (3.6). Out of the four roots of the obtained characteristic equations two have negative real parts. If the remaining two roots are to have positive real parts, it is necessary in conformity with (2.6) that the equalities

$$\lambda_r < \lambda_i^2/\omega^2, \quad \lambda_r < 1 + L\lambda_i^2/\omega^2 \tag{3.7}$$

be satisfied. If at least one of these inequalities is satisfied, system (3.1) has a solution emanating from $u = 0$.

Since in region $0 < u < \varepsilon$ the coefficients of the linear system (3.1) do not have singularities, its solution exist throughout that region /7/ and are integral analytic functions of parameter λ.

The form of solutions (3.5) is altered when the roots of the characteristic equations differ by an integer /7/, however, the pattern of integral curve behavior, i.e. the number of linearly independent solutions of (3.1) emanating from $u = 0$, is again determined by the signs of the real parts of these equations.

4. Analysis of the point spectrum of operator A.

We denote the solutions emanating from $u = 0$ by z_1^0 and z_2^0, and those emanating from $u = 1$ in region $\varepsilon < u < 1$ by z_1^1 and z_2^1. For z_1^0 and z_2^0 to be eigenfunctions of operator $\mathbf{A}$ it is necessary that their continuation into region $\varepsilon < u < 1$ vanish for $u = 1$. Such continuation is effected using Eq. (3.2) and the conditions for $u = \varepsilon$

$$z_k^0(\varepsilon) = z_k^1(\varepsilon), \quad \frac{dz_k^0(\varepsilon)}{du} = \frac{dz_k^1(\varepsilon)}{du}, \quad k = 1, 2 \tag{4.1}$$

which are a corollary of the assumption of continuity of $f(u)$ when $u = \varepsilon$.

The symmetric about the real axis part of region of eigenvalues in the λ-plane is shown in Fig.1. The respective equations of curves 1, 2, and 3 are of the form

$$\lambda_r = L\lambda_i^2/\omega^2, \quad \lambda_r = 1 + L\lambda_i^2/\omega^2, \quad \lambda_r = \lambda_i^2/\omega^2$$

When both inequalities (3.4) and the second of inequalities (3.7) (the shaded region in Fig.1), i.e. when

$$1 + L\lambda_i^2/\omega^2 > \lambda_r > \begin{cases} \lambda_i^2/\omega^2, & L < 1 \\ L\lambda_i^2/\omega^2, & L \geqslant 1 \end{cases} \tag{4.2}$$

the operation of continuation of solutions z_1^0 and z_2^0 into region $\varepsilon < u < 1$ yields a system of four linear equations in C_1, C_2, C_3, and C_4, which is uniquely solvable, thus resolving positively the question of continuation. All λ which satisfy inequalities (4.2) are eigenvalues.

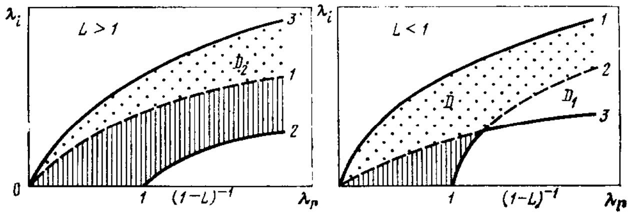

Fig.1

When $\lambda \in D$, i.e.

$$L\lambda_i^2/\omega^2 < \lambda_r < \min\{\lambda_i^2/\omega^2,\ 1 + L\lambda_i^2/\omega^2\},\quad L < 1$$

the two-parameter set z_2^1 and the one-parameter $z_1^1\,(C_2 = 0)$ emanate from $u = 1$. The constants C_3 and C_4 are uniquely determined.

For the determination of C_1 we have two equations

$$z_1^0(\varepsilon,\lambda) = C_1(1-\varepsilon)^{D_1},\quad \frac{dz_1^0(\varepsilon,\lambda)}{du} = -C_1D_1(1-\varepsilon)^{D_1-1}$$

which imply that the solution z_1^0 must satisfy for $u = \varepsilon$ the supplementary condition

$$(1-\varepsilon)\frac{dz_1^0(\varepsilon,\lambda)}{du} + D_1z_1^0(\varepsilon,\lambda) = 0 \tag{4.3}$$

The last equality can only be satisfied for a discrete set of λ (a denumerable set of eigenvalues).

The proof of existence of a denumerable set of eigenvalues in the case of

$$\lambda_r = \min(\lambda_i^2/\omega^2,\ 1 + L\lambda_i^2/\omega^2),\quad 1 + L\lambda_i^2/\omega^2 < \lambda_r < \lambda_i^2/\omega^2,\quad L < 1 \tag{4.4}$$

is similar (the dash line and region D_1 for $L < 1$ in Fig.1) also when $\lambda \in D_2$, i.e.

$$\lambda_i^2/\omega^2 < \lambda_r \leqslant L\lambda_i^2/\omega^2,\quad L > 1 \tag{4.5}$$

In the case of (4.4) with $u = \varepsilon$ (4.3) holds, while in the case of (4.5) $(C_4 = 0)$ the following condition must be satisfied

$$(1-\varepsilon)\frac{dz_2^0(\varepsilon,\lambda)}{du} + B_1z_2^0(\varepsilon,\lambda) = 0 \tag{4.6}$$

When

$$\lambda_r \geqslant \max\{\lambda_i^2/\omega^2,\ 1 + L\lambda_i^2/\omega^2\}$$

no integral curves emanate from $u = 0$ and, consequently, there are no eigenvalues.

The determination of stability or instability of stationary solutions of (1.1) is evidently associated with the analysis of eigenvalues in the region

$$\lambda_r \leqslant \begin{cases} L\lambda_i^2/\omega^2, & L < 1 \\ \lambda_i^2/\omega^2, & L \geqslant 1,\ \lambda \neq 0 \end{cases} \tag{4.7}$$

Such analysis is carried out below.

5. Stability in the case of $L = 1$.

Let condition (4.7) be satisfied. It follows from (2.1) that $v = u$ and $\Phi = \Phi(u)$. Subtracting in (3.1) the second equation from the first, we obtain

$$p^2(z_1 - z_2)'' + p(p' + \omega)(z_1 - z_2)' + \lambda(z_1 - z_2) = 0$$

whose solution for $z_1 - z_2$ is of the form

$$z_1 - z_2 = E_1\exp\left(\mu_1\int^u\frac{du}{p}\right) + E_2\exp\left(\mu_2\int^u\frac{du}{p}\right) \tag{5.1}$$

where E_1 and E_2 are constants of integration, and μ_1 and μ_2 are the roots of the equation $\mu^2 + \omega\mu + \lambda = 0$.

Taking into account the unboundedness of the right-hand side of (5.1) as $u \to 0$ or $u \to 1$ we obtain from it that $z_1 = z_2$. System (3.1) reduces to the single equation

$$p^2z_1'' + p(p' + \omega)z_1' + (a - b)z_1 = -\lambda z_1,\quad z_1(0) = z_1(1) = 0 \tag{5.2}$$

The characteristic equations for the analysis of solutions (5.2) in the neighborhood of $u = 0$ and $u = 1$ are, respectively, of the form

$$p_1^2\rho^2 + \omega p_1\rho + \lambda - 1 = 0,\quad D^2 - D + \lambda/\omega^2 = 0 \tag{5.3}$$

The case of multiple roots of Eqs. (5.3)

$$\lambda \neq 1 + \omega^2/4,\quad \lambda \neq \omega^2/4$$

is not considered here, since such values of λ lie in the stability region and are, consequently, of no interest.

As implied by (2.5) and (2.6), the real parts of the roots of Eqs. (5.3) have different signs. This means that one-parameter sets of curves emanates from $u = 0$ and $u = 1$. It is necessary to show that these are different sets.

Using the substitution

$$w = z_1\exp\left(\frac{\omega}{2}\int^u\frac{du}{p}\right) \tag{5.4}$$

we reduce Eq. (2.5) to the self-conjugate form

$$lw = -\lambda w, \quad l = p\frac{d}{du}\left(p\frac{d}{du}\right) - \frac{\omega^2}{4} + a - b \tag{5.5}$$

Transform (5.4) does not alter the pattern of behavior of integral curves of Eq. (5.2)in the neighborhood of $u = 0$ and $u = 1$, since in the neighborhood of $u = 0$, $z_1 \sim u^\rho$

$$\exp\left(\frac{\omega}{2}\int^u \frac{du}{p}\right) \sim u^{\omega/2p_1}, \quad w \sim u^{\rho+\omega/(2p_1)}$$

$$\rho + \frac{\omega}{2p_1} = \pm\sqrt{\frac{\omega^2 + 4(1-\lambda)}{4p_1^2}}$$

This implies that $|w| \to 0\,(\infty)$ if $|z_1| \to 0\,(\infty)$.

The proof for the neighborhood of $u = 1$ is similar.

Corollary. If the integral curves of Eq. (5.2) emanating from $u = 0$ and $u = 1$ belong to a single set, then the integral curves of Eq. (5.5) also belong to a single set.

The analysis of eigenvalues of problem (5.2) thus reduces to that of eigenvalues of operator l determinate over functions that vanish for $u = 0$ and $u = 1$.

Further proof is analogous to that in /6/. The self-conjugate operator l has the eigenfunction

$$w_0(u) = p\exp\left(\frac{\omega}{2}\int^u \frac{du}{p}\right)$$

which corresponds to $\lambda = 0$.

Since that eigenfunction has no zeros in the interval (0.1), it corresponds to the greater eigenvalue /8/. Hence the operator l has no negative eigenvalues.

Remark 1. The point spectrum for $L = 1$ contains not only pure real values, which is in disagreement with /6/. It was stated in /9/ that on the basis of /6/ the spectral region consists of points $\lambda = 0, \lambda \geqslant \omega^2/4$. The discrepancy between these conclusions and the results obtained here are explained by that the pattern of behavior of integral curves z_1 in the neighborhood of points $u = 0$ and $u = 1$ remains unaltered by transform (5.4) not for all λ, as was tacitly assumed in /6/.

Remark 2. The analysis of stability is possible not only in the case of the one-dimensional problem. For example, in the case of two-dimensional perturbations in the right-hand sides of each of Eqs. (1.1) we have, respectively, the terms $\partial^2 U/\partial y^2$ and $\partial^2 V/\partial y^2$), and instead of (1.6) we have to seek a solution of the linearized equations of the form

$$\varphi_1 = z_1(\xi, \lambda)\exp(-\lambda t + i\alpha y), \quad \psi = z_2(\xi, \lambda)\exp(-\lambda t + i\alpha y)$$

where α is the wave number. Using the obtained here results, we obtain for the curves bounding the point spectrum in the complex plane λ the equations

$$\lambda_r - L\alpha^2 = L\lambda_i^2/\omega^2, \quad \lambda_r - L\alpha^2 = 1 + L\lambda_i^2/\omega^2, \quad \lambda_r - \alpha^2 = \lambda_i^2/\omega^2$$

When $L > 0, \alpha \neq 0$ the spectral region moves in the direction of positive λ_r, i.e. into the region of stability.

REFERENCES

1. BAUSHEV, V. S. and VILIUNOV, V. N., Estimates of normal propagation velocities of laminar and small scale turbulent flames. PMTF, No.3, 1976.
2. ZEL'DOVICH, Ia. B., On the theory of flame propagation. Zh.Fiz.Khimii, Vol.22, No.1,1948.
3. KANEL', Ia. I., On the stationary solution for the system of equations of the theory of combustion. Dokl.Akad.Nauk SSSR, Vol.149, No.2, 1963.
4. LIN' CHIA-CHIAO, The Theory of Hydrodynamic Stability, Cambridge University Press, 1955.
5. IUDOVICH, V. I., On the stability of stationary flows of viscous incompressible fluid.Dokl. Akad.Nauk SSSR, Vol.161, No.5, 1965.
6. BARENBLATT, G. I. and ZEL'DOVICH, Ia. B., On the stability of flame propagation, PMM,Vol. 21, No.6, 1957.
7. KAMKE. E., Differentialgleichungen: Lösungsmethoden und Lösungen. Band 1. Gewöhnliche Differentialgleichungen. New York, Cnelsea Publ, Co., 1948.
8. TITCHMARSH, E. C., Expansions in Eigenfunctions Associated with Second Order Differential Equations, pt. 1 /Russian translation/. Moscow, Izd. Inostr. Lit., 1960.
9. ZEL'DOVICH, Ia. B., Theory of perturbation of equations with an invariance group, using flame propagation as an example. Dokl. Akad.Nauk, SSSR, Vol.230, No.3. 1976.

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